The Ackermann Approach for Modal Logic, Correspondence Theory and Second-Order Reduction: Extended Abstract

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Abstract

We introduce improvements for second-order quantifier elimination methods based on Ackermann's Lemma and investigates their application in modal correspondence theory. In particular, we define refined calculi and procedures for solving the problem of eliminating quantified propositional symbols from modal formulae. We prove correctness results and use the approach to compute first-order frame correspondence properties for modal axioms and modal rules. Our approach can solve two new classes of formulae with wider scope than existing classes known to be solvable by second-order quantifier elimination methods.

1 Second-Order Quantifier Elimination

An application of second-order quantifier elimination is correspondence theory in modal logic. Propositional modal logics, when defined axiomatically, have a second-order flavour, but can often be characterized by classes of model structures which satisfy first-order conditions. Frequently, with the help of second-order quantifier elimination methods, these first-order conditions, called frame correspondence properties, can be automatically derived from the axioms. For example, using the standard relational translation method the modal axiom **D** $= \forall p[\Box p \rightarrow \Diamond p]$ translates to this second-order formula:

$$\forall P \forall x[\forall y[R(x,y) \rightarrow P(y)] \rightarrow \exists z[R(x,z) \wedge P(z)]]. \quad (1)$$

This formula is equivalent to a first-order formula, namely $\forall x \exists y[R(x,y)]$, and is the first-order correspondence property of axiom **D**. It can be derived automatically with a second-order quantifier elimination method by eliminating the second-order quantifier $\forall P$ from (1).

Several second-order quantifier elimination methods exist. These methods belong to two categories: (i) substitution-rewrite approaches which exploit monotonicity properties, and (ii) saturation approaches, which are based on exhaustive deduction of consequences. Methods following the substitution-rewrite approach include the Sahlqvist-van Benthem substitution method for modal logic, the DLS algorithm introduced by Szalas in (1993) and together with Doherty and Lukaszewicz in (1997), the SQEMA algorithm for modal logic introduced by Conradie, Goranko and Vakarelov in (2006). Methods following the saturation approach include the SCAN algorithm of Gabbay and Ohlbach (1992), and hierarchical resolution of Bachmair, Ganzinger and Waldmann (1994).

Here, I am interested in methods using the substitution-rewrite approach to second-order quantifier elimination. In particular, my focus is on methods that are based on a general substitution property found in Ackermann (1935). This result, called *Ackermann's Lemma*, tells us when quantified predicate symbols are eliminable from second-order formulae. For propositional and modal logic Ackermann's Lemma can be formulated as follows. In any model,

$$\exists p[(\alpha \rightarrow p) \wedge \beta(p)] \quad \text{is equivalent to} \quad \beta^p_\alpha, \quad (2)$$

provided these two conditions hold: (i) p is a propositional symbol that does not occur in α, and (ii) p occurs only negatively in β. The formula β^p_α denotes the formula obtained from β by uniformly substituting α for all occurrences of p in β. This property is also true, when the polarity of p is switched, that is, all occurrences of p in β are positive and the implication in the left conjunct is reversed. Applied from left-to-right the equivalence (2) of Ackermann's Lemma eliminates the second-order quantifier $\exists p$. In fact, all occurrences of p are eliminated. This idea can be turned into an algorithm for eliminating existentially quantified propositional symbols. I refer to this algorithm as the *basic Ackermann algorithm*.

2 A Refined Ackermann Approach

Based on the basic Ackermann algorithm I introduce a refined second-order quantifier elimination approach for modal logic. Like the SQEMA algorithm, rather than translating the modal axiom into second-order logic and then passing it to a second-order quantifier elimination method, the approach performs second-order quantifier elimination directly in modal logic. Only in a subsequent step the translation to first-order logic is performed. For example, given the second-order modal formula $\forall p[\Box p \rightarrow \Diamond p]$, the approach first eliminates $\forall p$ from the formula and returns the formula $\Diamond \top$. Subsequently this is translated to first-order logic to give the expected seriality property $\forall x \exists y [R(x, y)]$.

The approach is defined for propositional multi-modal tense logics, more precisely, the logic $\mathbf{K}^n_{(m)}(\smile, \pi+)$ with forward and backward looking modalities, nominals, and second-order quantification over propositional symbols.

A main motivation for this work has been to gain a better understanding of when quantifier elimination methods succeed, and to pinpoint precisely which techniques are crucial for successful termination. I define two new classes of formulae for which the approach succeeds: the class $\mathcal{C}$ and an algorithmic version called $\mathcal{C}^>$. The classes define normal forms for when Ackermann-based second-order quantifier elimination methods succeed. $\mathcal{C}$ and $\mathcal{C}^>$ subsume both the Sahlqvist class of formulae and the class of monadic-inductive formulae of Goranko and Vakarelov (2006). I present minimal requirements for successful termination for all these classes. This allows existing results of second-order quantifier elimination methods to sharpened and strengthened.

I consider two applications of the approach:

(i) Computing correspondence properties for modal axioms *and modal rules*. For example, equivalently reducing axiom **D** to the seriality property, or equivalently reducing the modal rule $\Box p / \Diamond p$ to $\forall x \exists y \exists z [R(x, y) \wedge R(z, y)]$.

(ii) Equivalently reducing of second-order modal problems. For example, the second-order modal formula $\forall p \forall q [\Box(p \vee q) \rightarrow (\Box p \vee \Box q)]$ equivalent reduces to $\forall p [\Diamond p \rightarrow \Box p]$, or the axiom **D** equivalently reduces to $\Diamond \top$.

While the approach follows the idea of the basic Ackermann algorithm and is closely related to the DLS algorithm and the SQEMA algorithm, I introduce a variety of enhancements and novel techniques.

First, which propositional symbols are to be eliminated can be flexibly specified, and the approach is not limited to eliminating all propositional symbols. Second, in order to be able to ensure effectiveness and avoid unintended looping, the approach is enhanced with ordering refinements. In the approach an ordering on the non-base symbols (these are the symbols that we want to eliminate) must be specified and determines the order in which these symbols are eliminated. At the same time the ordering is used to delimit the way that the inference rules are applied. It turns out, that the adoption of ordering refinements allows for a more in-depth analysis of how the inferences are performed and a better understanding of the properties of the approach. Third, for reasons of efficiency, and improved success rate, it is beneficial to incorporate techniques for pruning the search space. A general notion of redundancy is thus included. It is designed so that it is possible to define practical simplification and optimization techniques in a flexible way. Fourth, the approach is defined in terms of calculi given by sets of inference rules. This has the advantage that the approach can be studied independently of practical issues such as rule application order, strategies and heuristics. It allows for a more fine-grained analysis of the computational behaviour of the approach and more general results can be formulated.

3 Results

The following results have been obtained.

1. Any derivation in the approach is guaranteed to terminate and the obtained formula is logically equivalent to the input formula. This means the refined modal Ackermann calculus is correct and terminating.

2. Any problem in the class $\mathcal{C}^>$ is effectively and successfully reducible by the rules of the approach using some ordering.

3. For the subclass $\mathcal{C}$ of $\mathcal{C}^>$, the sign switching rule, redundancy elimination are not needed, and the ordering is immaterial.

4. Whenever the approach successfully eliminates all propositional symbols for a modal formula α then (a) $\neg\alpha$ is d-persistent and hence canonical, and (b) the formula returned is equivalent to α.

5. All modal axioms equivalent to the conjunction of formulae reducible to clauses in $\mathcal{C}$ and $\mathcal{C}^>$ are elementary and canonical.

These results are improvements for substitution-rewrite approaches based on Ackermann's Lemma, and present strengthenings of Sahlqvist's theorem and the corresponding result for monadic-inductive formulae.

The significance of the last result is that axioms that are equivalent to first-order properties and are canonical can be used to provide sound and complete axiomatizations of modal logics.

4 Further details

For a full account of the approach and the results, I refer to Schmidt (2008) and Chapter 13 in Gabbay et al. (2008).

References

D. M. Gabbay, R. A. Schmidt, and A. Szałas. *Second-Order Quantifier Elimination: Foundations, Computational Aspects and Applications*. College Publications, 2008.

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